Real Analysis Lecture (Sem2)

TS

Theorem Suppose for each $n \in \mathbb{N}$, A_n is denumerable. Then $\bigcup_{n=1}^{\infty} A_n$ is denumerable.

Proof: To show $\bigcup_{n=1}^{\infty} A_n$ is denumerable, it is sufficient to produce a one to one map $f: \bigcup_{n=1}^{\infty} A_n \to \mathbb{N}$ (By last day lecture)

We begin with showing that we can find pairwise disjoint countable sets (either finite or Denumerable) B_n s.t $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$

~~Before~~ Remember the last day lecture where we show union of two countable set is countable

we define
$$B_1 = A_1$$
$$B_2 = A_2 \setminus A_1$$
$$B_3 = A_3 \setminus (A_1 \cup A_2)$$
$$\vdots$$
$$B_n = A_n \setminus (A_1 \cup A_2 \cup \dots \cup A_{n-1})$$
$$\vdots$$

To see that B_i's are pairwise disjoint:

Consider B_m and B_n with $m < n$. let $x \in B_m$

$\Rightarrow x \in A_m \setminus (A_1 \cup A_2 \cup \dots \cup A_{m-1})$

$\Rightarrow x \notin A_n \setminus (A_1 \cup A_2 \cup \dots \cup A_m \cup \dots \cup A_{n-1})$

$\Rightarrow x \notin B_n$. Therefore $B_m \cap B_n = \phi$

Now we prove that $\bigcup_{n=1}^{\infty} B_n \overset{TS}{=} \bigcup_{n=1}^{\infty} A_n$.

Since $B_n = A_n \setminus (A_1 \cup A_2 \cup \dots \cup A_{n-1}) \subseteq A_n$ for all n.

$\Rightarrow \bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n=1}^{\infty} A_n$ —— (i)

Conversely let $x \in \bigcup_{n=1}^{\infty} A_n \Rightarrow x \in A_n$ for some $n \in \mathbb{N}$

Pick smallest n_0 s.t $x \in A_{n_0} \Rightarrow x \in B_{n_0}$

as $x \notin A_1 \cup A_2 \cup \dots \cup A_{n_0-1}$

Therefore $x \in \bigcup_{n=1}^{\infty} B_n \Rightarrow \bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} B_n$ —— (ii)

Using (i) and (ii) we have.

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n.$$

Since $B_n \subseteq A_n$ for all n. Therefore, each B_i's are either finite or Denumerable. Thus there exist a one to one function $f_n : B_n \to \mathbb{N}$ for each n.

Now Consider the prime numbers list.

$p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ... p_n.

Now define a function $f : \bigcup_{n=1}^{\infty} B_n \to \mathbb{N}$ as follows

$$f(x) = p_n^{f_n(x)} \text{ if } x \in B_n.$$

For example: if $x \in B_3$ then $f_3 : B_3 \to \mathbb{N}$, $f_3(x)$ is a natural number say $f_3(x) = 8$. then $f(x) = p_3^{f_3(x)} = 5^8$.

Show that $f : \bigcup_{n=1}^{\infty} B_n \to \mathbb{N}$ is one to one. ?

(Remember the last class)

Thus f is one to one. Thus $\bigcup_{n=1}^{\infty} B_n$ is denumerable

i.e $\bigcup_{n=1}^{\infty} A_n$ is denumerable. (Proved)

Application : Prove that the set of all positive rational numbers is a denumerable set.

positive rational numbers are $\mathbb{Q}^+ = \{ \frac{p}{q} : p, q \in \mathbb{N} \}$

Example : $\{ \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \ldots \} \subseteq \mathbb{Q}^+$

$\{ \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \ldots \} \subseteq \mathbb{Q}^+$

$\{ \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \ldots \} \subseteq \mathbb{Q}^+$

Proof Define for each $n \in \mathbb{N}$

$$A_n = \{ \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots \}$$

Each A_n is denumerable since

$f_n : \mathbb{N} \to A_n$

$f_n(i) = \frac{i}{n}$ is one to one and onto.

Now $\mathbb{Q}^+ = \bigcup_{n=1}^{\infty} A_n$.

Thus $\mathbb{Q}^+$ is the countable union of denumerable sets.

then $\mathbb{Q}^+$ is denumerable.

Application 2 : Set of all negative rational numbers is denumerable.

$$Q^- = \left\{ -\frac{p}{q} : p, q \in \mathbb{N} \right\}$$

we proceed as earlier

Define for each n.

$$B_n = \left\{ -\frac{1}{n}, -\frac{2}{n}, -\frac{3}{n}, \dots \right\}$$

Each B_n is denumerable since the map

$$f_n : \mathbb{N} \to B_n.$$

$$f_n(i) = -\frac{i}{n}. \quad \text{is a bijection.}$$

Now $Q^{-} = \bigcup_{n=1}^{\infty} B_n$, thus Q^{-} is the countable union of denumerable sets. thus Q^- is denumerable.

Application 3 Set of all rational numbers is denumerable.

Now $Q = Q^- \cup \{0\} \cup Q^+$

$Q^- \to$ denumerable

$Q^+ \to$ denumerable.

Thus $Q^- \cup Q^+ \to$ denumerable.

$\{0\}$ is a finite set. Thus $Q^+ \cup \{0\} \cup Q^- \to$ denumerable